Electricity and Magnetism

Third Semester BSc Physics

Pondicherry University

Sandeep K V, MSc NET
Assistant Professor
Department of Physics
Mahatma Gandhi Government Arts College, Mahe
U.T of Puducherry

kvsandeep07@gmail.com
kvsandeep07@py.gov.in
sandeepkv@mggacmahe.ac.in
www.mggacmahe.ac.in

9447546957

November 29, 2022

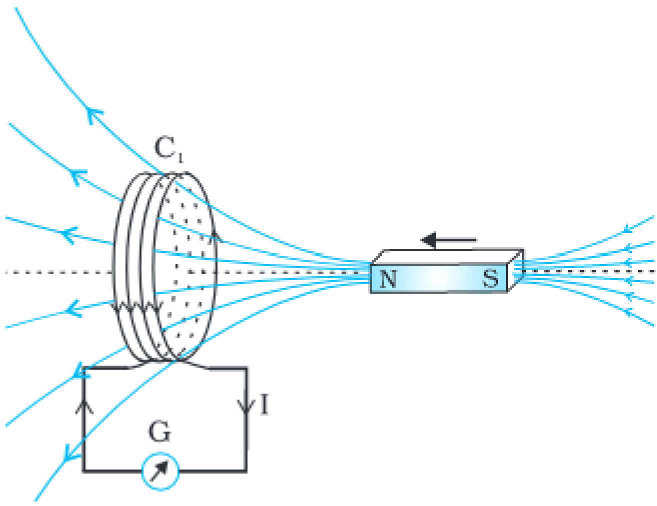
Contents

Chapter 4

4.1 Electromagnetic Induction

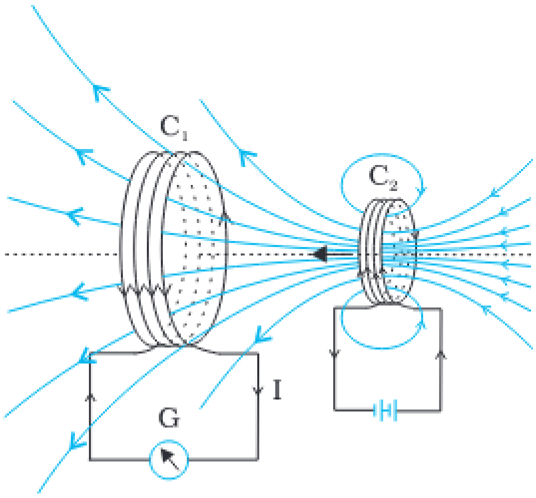
The phenomenon in which electric current is generated by varying magnetic fields is called electromagnetic induction. Whenever magnetic lines of force are cut by a closed conducting circuit, an induced current flows in the circuit which lasts only so long as the flux is changing. The emf which produces the induced current is called induced emf. induction. The discovery and understanding of electromagnetic induction are based on several experiments carried out by Faraday and Henry.

When the bar magnet is pushed towards the coil, the pointer in the galvanometer G gets deflected.



The experiment shows that it is the relative motion between the magnet and the coil that is responsible for generation (induction) of electric current in the coil.

Current is induced in coil C_1 due to motion of the current carrying coil C_2. The relative motion between the coils induces the electric current.



Through another experiment, Faraday showed that the relative motion is not an absolute requirement. Whenever current in the primary coil is increased or decreased, then also emf is induced in the secondary coil. Also emf is induced in the secondary coil at the time of make and break of the key in the primary coil.

4.2 Faraday's laws of electromagnetic induction

From the experimental observations, Faraday arrived at a conclusion that an emf is induced in a coil when magnetic flux through the coil changes with time. Faraday stated experimental observations in the form of a law called Faraday's law of electromagnetic induction.

4.2.1 First Law

Whenever the magnetic flux through a conductor is changed, an emf is induced in the conductor. An emf is induced in a coil when magnetic flux through the coil changes with time.

4.2.2 Second Law

The magnitude of the induced emf in a circuit is equal to the time rate of change of magnetic flux through the circuit.
If ϕ is the magnetic flux linked with the circuit at any instant t and ϵ is the induced e.m.f., then,

$$\epsilon = \frac{d\phi}{dt} \tag{4.1}$$

The direction of the induced emf, or current, is such as to oppose the change that produced it. This is also known as **Lenz's law**. Combining this with the Faraday's laws,

$$\epsilon = -\frac{d\phi}{dt} \tag{4.2}$$

The induced emf can be increased by increasing the number of turns N of a closed coil. For a coil of **N** turns, the above equation reads,

$$\epsilon = -N\frac{d\phi}{dt} \tag{4.3}$$

4.2.3 Integral and Differential forms of Faraday's law

Consider a closed circuit **S** with its boundary **C**. If **dS** is a small area of it and **B** the total magnetic field linked with it, then the total flux through it

$$\phi = \int_S B \cdot dS \tag{4.4}$$

When the magnetic flux is changed, an electric field **E** is induced around the circuit, as per Faraday's law. The line integral of the electric field round the circuit gives the induced emf.

$$\epsilon = \oint_C E \cdot dl \tag{4.5}$$

According to Faraday's law, the emf induced in an electric circuit is equal to the negative of the time rate of change of the magnetic flux linked with the circuit.

$$\epsilon = -\frac{d\phi}{dt} = -\frac{d}{dt}\int_S B \cdot dS \tag{4.6}$$

From equations 4.5 and 4.6, we get

$$\oint_C E \cdot dl = -\frac{d}{dt}\int_S B \cdot dS \tag{4.7}$$

Equation 4.7 is the **integral form of the Faradays law**. It states that the line integral of the electric field around any closed circuit is equal to the negative time-rate of change of the magnetic flux through the circuit.

When **B** changes with time and circuit is fixed, then

$$-\frac{d}{dt}\int_S B \cdot dS = -\int_S \frac{\partial B}{\partial t} \cdot dS \tag{4.8}$$

But using Stoke's theorem,

$$\oint_C E \cdot dl = \int_S (\nabla \times E) \cdot dS \tag{4.9}$$

Combining equations 4.8 and 4.9, we get

$$\int_S (\nabla \times E) \cdot dS = -\int_S \frac{\partial B}{\partial t} \cdot dS \tag{4.10}$$

From equation 4.10,

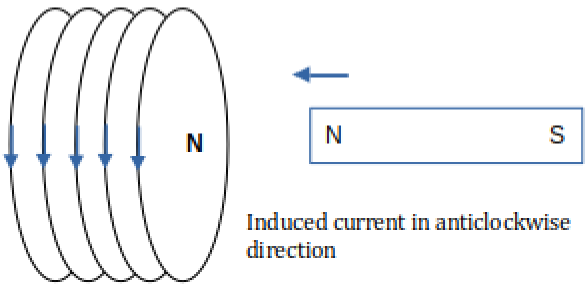
$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{4.11}$$

Equation 4.11 is the differential form of Faraday's law of electromagnetic induction. This is one of the four **Maxwell equations in electrodynamics**.

4.2.4 Lenz's law

Lenz's law gives the direction of the induced current and it is based on the law of conservation of energy. *Lenz's law, in electromagnetism, states that an induced electric current flows in a direction such that the current opposes the change that induced it.*
The direction of induced current is such that it opposes the cause that produces it. Thus there is extra work done against the opposing force. The work done against the opposing force results

in the change in the magnetic flux and hence the current is induced. This extra work done is the electrical energy, and hence the law of conservation of energy is satisfied.

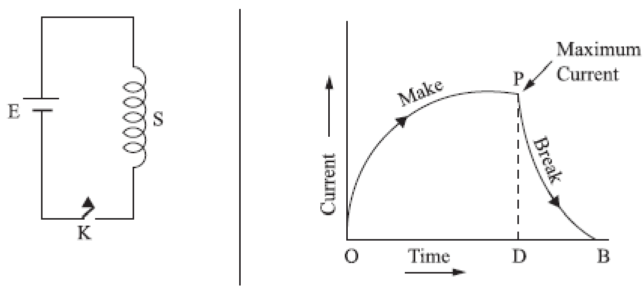


When the North pole of a bar magnet is taken in to a circular coil, an emf is induced in it. The induced current makes a coil an equivalent bar magnet. Then according to Lenz's, at the face of the coil to which North pole of the bar magnet is taken, the induced current should flow in the anticlockwise direction so that a North pole, which repels the North pole of the bar magnet, is produced at that face of the coil due to the induced current.

4.3 Self Induction

The phenomenon of the production of an induced e.m.f. in a circuit itself due to the change in current through it is called self induction and the induced e.m.f. is called back e.m.f.

When a current flows in a coil, a magnetic field is set up in it and if the current through the coil is changed, the flux linked with the coil also changes. An induced emf is set up in the coil. But by Lenz's law, the direction of induced emf is such as to oppose the change in current. When the current is increasing, the induced emf is against the current. When the current is decreasing, the induced emf is in the direction of current. This phenomenon is called **self-induction**. The induced emf is called **back emf**.



The self-induced emf in the coil will resist the rise of current when the current increases and it will also resist the fall of current if the current decreases which means that the direction of the induced emf is opposite to the applied voltage if the current is increasing and the direction of the induced emf is in the same direction as the applied voltage if the current is falling.

4.3.1 Self inductance

The magnetic flux ϕ produced in a coil due to a current I is directly proportional to the current I flowing in the coil.

$$\phi \propto I \tag{4.12}$$

Or

$$\phi = LI \tag{4.13}$$

where L is the constant of proportionality, called the **coefficient of self induction or self inductance** of the coil. It is the total magnetic flux linked with it when a unit current passes through it.

Then according to Faraday's law, the self induced emf,

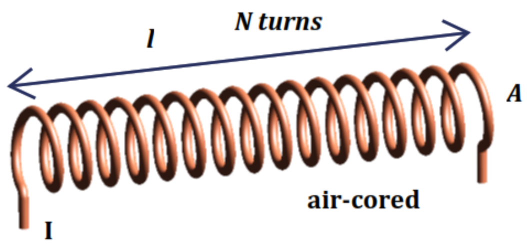
$$\epsilon = -\frac{d\phi}{dt} = -L\frac{dI}{dt} \tag{4.14}$$

Thus Self-inductance of a coil is numerically equal to the induced emf when current in it is changing at unit rate. The S.I. unit of self-inductance is **henry (H)**.

The inductance plays the same role in a circuit as mass and *moment of inertia* play in mechanical motion. When a circuit is switched on, the increasing current induces an emf which opposes the growth of current in a circuit. Likewise, when circuit is broken, the decreasing current induces an emf in the reverse direction. This emf now opposes the decay of current. Thus, inductance of the coil opposes any change in current and tries to maintain the original state.

4.3.2 Self-Inductance of a long solenoid

Consider an air-cored long solenoid of **N** turns with length **l** and area of cross section **A** carrying a current **I**.



If **B** is the magnetic field at any point inside the solenoid, then magnetic flux per turn

$$\frac{\phi}{N} = BA \tag{4.15}$$

But, we know, for a long solenoid, field inside

$$B = \frac{\mu_0 NI}{l} \tag{4.16}$$

Therefore, using equation 4.15, we get total magnetic flux ϕ linked with the solenoid

$$\phi = \frac{\mu_0 NIA}{l} \times N = \frac{\mu_0 N^2 IA}{l} \tag{4.17}$$

But, for a coil of self-inductance **L**, we have

$$\phi = LI \tag{4.18}$$

Comparing equations 4.17 and 4.18, we get

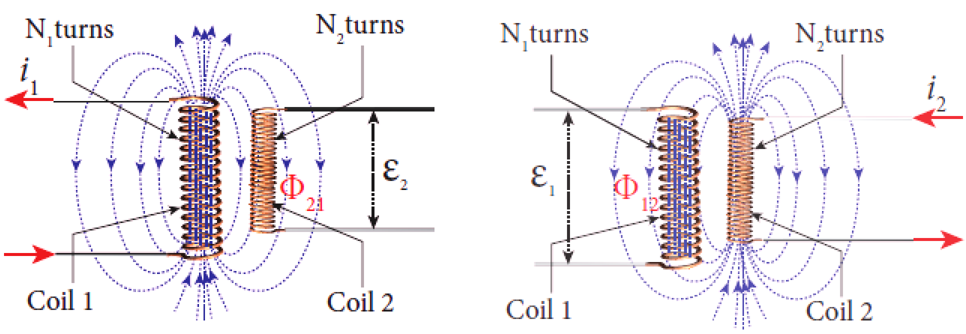
$$L = \frac{\mu_0 N^2 A}{l} \tag{4.19}$$

If the core is filled with a magnetic material of permeability μ,

$$L = \frac{\mu N^2 A}{l} = \frac{\mu_0 \mu_r N^2 A}{l} \tag{4.20}$$

4.4 Mutual Induction

When two coils are brought in proximity with each other, the magnetic field in one of the coils tend to link with the other and this leads to the generation of voltage in the second coil. This property of a coil which affects or changes the current and voltage in a secondary coil is called mutual induction between the two coils.
When an electric current passing through a coil changes with time, an emf is induced in the neighbouring coil. This phenomenon is known as mutual induction and the emf is called mutually induced emf. Mutual inductance is the main operating principle of generators, motors, and transformers.



Consider two coils which are placed close to each other. If an electric current I_1 is sent through coil 1, the magnetic field produced by it is also linked with coil 2. Let ϕ_{21} be the magnetic flux linked with each turn of the coil 2 of N_2 turns due to coil 1, then the total flux linked with coil 2 $(N_2\phi_{21})$ is proportional to the current I_1 in the coil 1.

$$\begin{aligned} N_2\phi_{21} &\propto I_1 \\ N_2\phi_{21} &= M_{21} I_1 \\ M_{21} &= \frac{N_2\phi_{21}}{I_1} \end{aligned} \tag{4.21}$$

The constant of proportionality M_{21} is the **mutual inductance of the coil 2 with respect to coil 1**. It is also called as **coefficient of mutual induction**. If $I_1 = 1\ A$, then $M_{21} = N_2\phi_{21}$.

Therefore, the mutual inductance M_{21} is defined as the flux linkage of the coil 2 when 1 A current flows through coil 1.
When the current I_1 changes with time, an emf ϵ_2 is induced in coil 2. From Faraday's law of electromagnetic induction, this mutually induced emf ϵ_2 is given by

$$\epsilon_2 = -\frac{d(N_2\phi_{21})}{dt} = -\frac{d(M_{21}I_1)}{dt}$$
$$\epsilon_2 = -M_{21}\frac{dI_1}{dt} \tag{4.22}$$
$$M_{21} = -\frac{\epsilon_2}{\frac{dI_1}{dt}}$$

The negative sign in the above equation shows that the mutually induced emf always opposes the change in current I_1 with respect to time.
Mutual inductance M_{21} is also defined as the opposing emf induced in the coil 2 when the rate of change of current through the coil 1 is 1 As^{-1}.
Similarly, if an electric current I_2 through coil 2 changes with time, then emf ϵ_1 is induced in coil 1. Therefore,

$$M_{12} = \frac{N_1\phi_{12}}{I_2}$$
$$M_{12} = -\frac{\epsilon_1}{\frac{dI_2}{dt}} \tag{4.23}$$

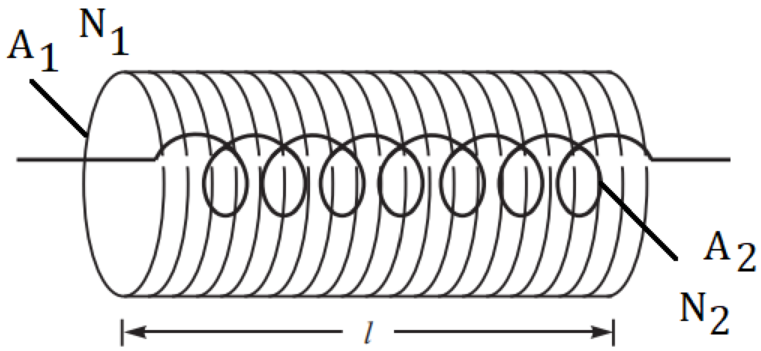
where M_{12} is the mutual inductance of the coil 1 with respect to coil 2. **It can be shown that for a given pair of coils, the mutual inductance is same.**

$$M_{12} = M_{21} = M \tag{4.24}$$

In general, the mutual induction between two coils depends on size, shape, the number of turns of the coils, their relative orientation and permeability of the medium.
The unit of mutual inductance is also henry (H).

4.4.1 Mutual inductance between two long co-axial solenoids

Consider two long co-axial solenoids of same length **l**. The length of these solenoids is large when compared to their radii so that the magnetic field produced inside the solenoids is uniform. Let A_1 and A_2 be the area of cross section of the solenoids with A_1 being greater than A_2. The turn density of these solenoids are $n_1 = N_1/l$ and $n_2 = N_2/l$ respectively.



Let I_1 be the current flowing through solenoid 1, then the magnetic field produced inside it is

$$B_1 = \frac{\mu_0 N_1 I_1}{l} \tag{4.25}$$

As the field lines of B_1 are passing through the area bounded by solenoid 2, the magnetic flux is linked with each turn of solenoid 2 due to solenoid 1 and is given by

$$\phi_{21} = B_1 A_2 = \frac{\mu_0 N_1 I_1 A_2}{l} \tag{4.26}$$

The flux linkage of solenoid 2 with total turns N_2 is

$$N_2 \phi_{21} = N_2 \frac{\mu_0 N_1 I_1 A_2}{l} \tag{4.27}$$

Comparing equations 4.21 and 4.27,

$$M_{21} = \frac{\mu_0 N_1 N_2 A_2}{l} = \mu_0 n_1 n_2 A_2 l \tag{4.28}$$

This gives the expression for **mutual inductance** M_{21} of the solenoid 2 with respect to solenoid 1.

Similarly, we can find mutual inductance M_{12} of solenoid 1 with respect to solenoid 2. The magnetic field produced by the solenoid 2 when carrying a current I_2 is $B_2 = \frac{\mu_0 N_2 I_2}{l}$. The magnetic field B_2 is uniform inside the solenoid 2 but outside the solenoid 2, it is almost zero. Therefore for solenoid 1, the area A_2 is the effective area over which the magnetic field B_2 is present; not area A_1. Thus we get **mutual inductance** M_{12} of the solenoid 1 with respect to solenoid 2, as

$$M_{12} = \frac{\mu_0 N_1 N_2 A_2}{l} = \mu_0 n_1 n_2 A_2 l \tag{4.29}$$

Thus

$$M_{12} = M_{21} = M = \frac{\mu_0 N_1 N_2 A_2}{l} = \mu_0 n_1 n_2 A_2 l \tag{4.30}$$

4.5 Energy stored in magnetic field

Consider an electric circuit containing inductance coil. When the circuit is closed, a back e.m.f. is induced in the circuit which opposes the growth of current in it.So a certain amount of work has to be done by the current in increasing the current from zero to a maximum value against the induced back e.m.f. The work done by the current is stored in the *magnetic field of the coil as potential energy*. When the circuit is switched off, an e.m.f. is induced in the opposite direction which opposes the decay of current. Thus the work done by the current during the growth is recovered.

Let **I** be the current in the inductor **L** at any instant **t**. Let the rate of growth of current be $\frac{dI}{dt}$. The back e.m.f. induced in the inductor is

$$\epsilon = -L\frac{dI}{dt} \tag{4.31}$$

and the work done in moving charge **dq** against this e.m.f. is

$$dW = -\epsilon dq = -L\frac{dI}{dt}dq = -L\frac{dq}{dt}dI = -LIdI \tag{4.32}$$

Total work done (energy stored in the inductor) in increasing the current from zero to maximum value I_0 is

$$U = W = \int_0^{I_0} -LIdI = \frac{1}{2}LI_0^2 \tag{4.33}$$

4.5.1 Energy density

Consider a very long air cored solenoid of length **l**, cross-sectional area **A**, and total number of turns **N**. Let I be the current in the solenoid. Magnetic Induction along axis inside the solenoid,

$$B = \frac{\mu_0 NI}{l} \tag{4.34}$$

and hence

$$I = \frac{Bl}{\mu_0 N} \tag{4.35}$$

The self-inductance of a solenoid

$$L = \frac{\mu_0 N^2 A}{l} \tag{4.36}$$

Substituting equations 4.35 and 4.36 in 4.33, we get the **energy stored in the magnetic field**

$$U = \frac{1}{2\mu_0}B^2 Al \tag{4.37}$$

Energy stored per unit volume or the **energy density u** in the magnetic field is

$$u = \frac{U}{\text{volume}} = \frac{U}{Al} = \frac{1}{2}\frac{B^2}{\mu_0} \tag{4.38}$$

4.6 Equation of continuity of current

The current density **J** at any point is defined as the quantity of charge passing per second through a unit area taken perpendicular to the direction of the flow of charge at that point.

Current **i** is related to the current density **J** by

$$i = \oint J.dS \tag{4.39}$$

where **dS** is an element of area and the integral is taken over any surface cutting across the conductor. Current density is a vector quantity.
Consider a closed surface **S** enclosing a region of volume **V**. Let the surface enclose some charge within it. Let $\boldsymbol{\rho}$ be the charge density (charge per unit volume) of the charge enclosed within the surface at any instant. Then the total charge $\boldsymbol{q}$ enclosed at any instant is given by

$$q = \int_V \rho dV \tag{4.40}$$

But, the current **i** is the rate of decrease of charge inside the volume,

$$i = -\frac{dq}{dt} = -\frac{d}{dt}\int_V \rho dV = -\int_V \frac{d\rho}{dt}dV \tag{4.41}$$

Comparing equations 4.39 and 4.41,

$$\oint J.dS = -\int_V \frac{d\rho}{dt}dV \tag{4.42}$$

But, According to Gauss divergence theorem,

$$\oint J.dS = \int_V (\nabla \cdot \boldsymbol{J})dV \tag{4.43}$$

Therefore, equation 4.42 becomes

$$\int_V (\nabla \cdot \boldsymbol{J})dV = -\int_V \frac{d\rho}{dt}dV \tag{4.44}$$

From equation 4.44,

$$\nabla \cdot \boldsymbol{J} + \frac{d\rho}{dt} = 0 \tag{4.45}$$

Equation 4.45 is called the **equation of continuity** and it expresses the fact that the charge is conserved.

4.7 Displacement current and modified Ampere's circuital law

Maxwell pointed out the mistake in Ampere's circuital theorem in some situations, to correct which he introduced the new concept of **displacement current**. *Faraday discovered that a changing magnetic field produces an electric field. Maxwell pointed out that a changing electric field produces a magnetic field.*

Consider a circuit of charging the parallel plate capacitor. If we apply Ampere's circuital theorem to find the magnetic field at a point such as P, in a region outside the capacitor, using an Amperian loop as shown in Fig a), then the theorem as given by the equation (Ampere's circuital law) below is correct.

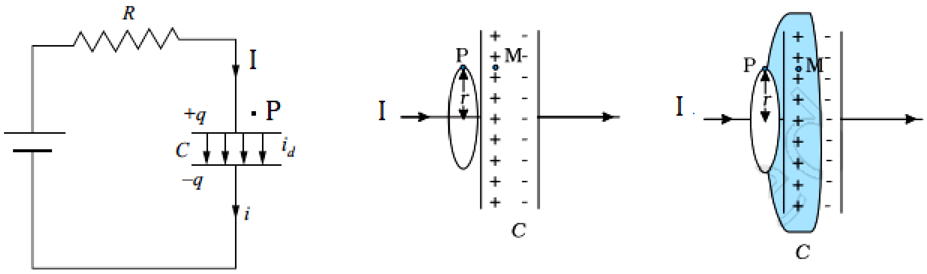


Fig a) Fig b)

$$\oint B \cdot dl = \mu_0 I \tag{4.46}$$

where **I** is the conduction current.

But if we consider an Amperian loop as shown in Fig b), there is no conduction current between the plates of the capacitor, and so the LHS of equation becomes zero. So we have a contradiction; calculated one way, there is a magnetic field at a point P; calculated another way, the magnetic field at P is zero. Since the contradiction arises from our use of Ampere's circuital law, this law must be missing something. The missing term must be such that one gets the same magnetic field at point P, no matter what surface is used.

But according to Maxwell, the **changing electric field between the plates** serves the purpose of conduction current inside the gap. The **displacement current** in the gap is found to be equal to the conduction current in the lead wires. This proves that the flow of current in a circuit is continuous. So we can rewrite the equation 4.46 as

$$\oint B \cdot dl = \mu_0 (I_C + I_d) \tag{4.47}$$

where $\boldsymbol{I_C} = I$ is the conduction current in the lead wires which is zero between the plates of the capacitor and $\boldsymbol{I_d}$ is the **displacement current** due to the changing electric flux between the capacitor plates which is zero outside the plates.

We have from Gauss' law in electrostatics,

$$\phi_E = \frac{q}{\epsilon_0} \tag{4.48}$$

where ϕ_E is the electric flux, between the plates, which varies with time. On charging/discharging the capacitor, the charge **q** in the capacitor varies with time. Then

$$\frac{d\phi_E}{dt} = \frac{1}{\epsilon_0}\frac{dq}{dt} = \frac{1}{\epsilon_0} I_d \tag{4.49}$$

Thus the magnitude of **displacement current is**

$$I_d = \epsilon_0 \frac{d\phi_E}{dt} \tag{4.50}$$

4.8 Maxwell's equations

There are **four fundamental equations of electromagnetism** known as Maxwell's equations.

4.8.1 In vacuum or free space

For fields in **vacuum**, in the presence of electric charge of density $\boldsymbol{\rho}$, and electric current of density **J**, the Maxwell's equations take the **differential** form

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} \quad \text{(Gauss law in electrostatics)} \tag{4.51}$$

$$\nabla \cdot B = 0 \quad \text{(Gauss law in magnetism)} \tag{4.52}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{(Faraday's law)} \tag{4.53}$$

$$\nabla \times B = \mu_0 J \quad \text{(Ampere's circuital law)} \tag{4.54}$$

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \quad \text{(Maxwell modified Ampere's circuital law)} \tag{4.55}$$

4.8.1.1 Derivation of Maxwell modified Ampere's circuital law

We have, from modified Ampere's circuital law,

$$\oint B \cdot dl = \mu_0 (I_C + I_d) = \mu_0 I_C + \mu_0 \epsilon_0 \frac{\partial \phi_E}{\partial t} \tag{4.56}$$

But since $\phi_E = \int E \cdot dS$, above equation becomes

$$\oint B \cdot dl = \mu_0 I_C + \mu_0 \epsilon_0 \int \frac{\partial E}{\partial t} \cdot dS \tag{4.57}$$

But according to Stoke's theorem, the LHS of equation 4.57 becomes

$$\oint B \cdot dl = \int_{surface} (\nabla \times B) \cdot dS \tag{4.58}$$

We have

$$I_C = \int_{surface} J \cdot dS \tag{4.59}$$

Using equations 4.58 and 4.59 in 4.57, we get

$$\int_{surface} (\nabla \times B) \cdot dS = \mu_0 \int_{surface} J \cdot dS + \mu_0 \epsilon_0 \int_{surface} \frac{\partial E}{\partial t} \cdot dS \tag{4.60}$$

From equation 4.60, we get the differential form of modified Ampere's circuital law, as

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 J + \mu_0 \frac{\partial D}{\partial t} \tag{4.61}$$

4.8.2 In linear isotropic dielectric medium

For a **linear, isotropic and homogeneous medium**,

$$\begin{aligned} D &= \epsilon E \\ B &= \mu H \\ J &= \sigma E \end{aligned} \tag{4.62}$$

Then the Maxwell equations can be written as

$$\nabla \cdot D = \rho \qquad \text{(Gauss law in electrostatics)} \tag{4.63}$$

$$\nabla \cdot B = 0 \qquad \text{(Gauss law in magnetism)} \tag{4.64}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{(Faraday's law)} \tag{4.65}$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} \qquad \text{(modified Ampere's circuital law)} \tag{4.66}$$

Where

$\rho \rightarrow$ charge density
$J \rightarrow$ current density
$E \rightarrow$ electric flux density
$B \rightarrow$ magnetic flux density (magnetic induction)
$D \rightarrow$ electric displacement vector
$H \rightarrow$ magnetic field intensity (or magnetising field vector)
$\sigma \rightarrow$ conductivity
$\mu \rightarrow$ permeability of the medium
$\epsilon \rightarrow$ permittivity of the medium

4.9 Poynting vector, S

When em waves propagate through matter from their source to distant point, there is a transfer of energy from source to that point. *The rate at which energy is transmitted through unit area perpendicular to the direction of propagation of energy is called Poynting vector (**The energy per unit time, per unit area, transported by the fields is called the Poynting vector**).* Mathematically, the Poynting vector S is given by

$$S \equiv \frac{1}{\mu_0}(E \times B) = (E \times H) \tag{4.67}$$

Poynting's theorem says, that the work done ($\frac{dW}{dt}$) on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface.

$$\frac{dW}{dt} = -\frac{d}{dt}\left(\frac{\epsilon_0}{2}E^2 + \frac{1}{\mu_0}B^2\right)\partial\tau - \frac{1}{\mu_0}\oint(E \times B) \cdot dS \tag{4.68}$$

Poynting's theorem is the '**work-energy theorem**' of electrodynamics. The firs integral on the right is the total energy stored in the fields. The second term evidently represents the rate at which energy is transported out of the volume, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface. The energy per unit time, per unit area, transported by the fields is called the Poynting vector.

4.10 Energy density in electromagnetic field

The work necessary to assemble a static charge distribution (against the Coulomb repulsion of like charges) is

$$W_E = \frac{\epsilon_0}{2}\int E^2 \partial\tau \tag{4.69}$$

where **E** is the resulting electric field. Likewise, the work required to get currents going (against the back emf) is

$$W_B = \frac{1}{2\mu_0}\int B^2 \partial\tau \tag{4.70}$$

where **B** is the resulting magnetic field.

This suggests that the total energy stored in electromagnetic fields, per unit volume (or **the energy density**), is

$$u = \frac{W_E + W_B}{\int \partial\tau} = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2 \tag{4.71}$$

4.11 Propagation of Electromagnetic waves in free space or vacuum

In regions of space where there is no charge ($\rho = 0$) or current ($J = 0$), Maxwell's equations take the form,

$$\nabla \cdot E = 0 \tag{4.72}$$

$$\nabla \cdot B = 0 \tag{4.73}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{4.74}$$

$$\nabla \times B = \mu_0\epsilon_0\frac{\partial E}{\partial t} \tag{4.75}$$

Taking curl of $\nabla \times E$, we get

$$\begin{aligned}\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E &= -\nabla \times \frac{\partial B}{\partial t} \\ &= -\frac{\partial}{\partial t}(\nabla \times B) = -\mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2}\end{aligned} \tag{4.76}$$

Similarly taking curl of $\nabla \times B$, we get

$$\begin{aligned}\nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B &= \nabla \times \mu_0\epsilon_0\frac{\partial E}{\partial t} \\ &= \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla \times E) = -\mu_0\epsilon_0\frac{\partial^2 B}{\partial t^2}\end{aligned} \tag{4.77}$$

Since $\nabla \cdot E = 0$ and $\nabla \cdot B = 0$, equations 4.76 and 4.77 becomes

$$\begin{aligned}\nabla^2 E = \mu_0\epsilon_0\frac{\partial^2 E}{\partial t^2} \\ \nabla^2 B = \mu_0\epsilon_0\frac{\partial^2 B}{\partial t^2}\end{aligned} \tag{4.78}$$

Equations 4.78 says that, in vacuum, then, each Cartesian component of **E** and **B** satisfies the three-dimensional wave equation,

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \tag{4.79}$$

So Maxwell's equations imply that empty space supports the propagation of electromagnetic waves, traveling at a speed of

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 2.998 \times 10^8 \ \ m/s \tag{4.80}$$

which is the velocity of light, **c**. This implies that light is an electromagnetic wave.

4.12 Transverse nature of electromagnetic wave

In an electromagnetic wave, electric and magnetic field vectors are perpendicular to each other and at the same time are perpendicular to the direction of propagation of the wave. This nature of electromagnetic wave is known as **transverse nature**.

Maxwell proved that both the electric and magnetic fields are perpendicular to each other in the direction of wave propagation.

In a region empty of electric charge, we have, from Maxwell's equations,

$$\begin{aligned} \nabla \cdot E &= 0 \\ \nabla \cdot B &= 0 \end{aligned}$$

Rewriting these equations

$$\begin{aligned} \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \end{aligned} \tag{4.81}$$

Now, consider an electromagnetic wave propagating in the $\boldsymbol{z}$ *direction* so that the space and time variation of the field components are of the form $\mathbf{cos}(\boldsymbol{kz - \omega t})$. The partial derivative of $\mathbf{cos}(\boldsymbol{kz - \omega t})$ with respect to x and y is zero

Since the spatial variation is zero in the x and y directions, equations 4.81 become

$$\begin{aligned} \frac{\partial E_z}{\partial z} = 0 \\ \frac{\partial B_z}{\partial z} = 0 \end{aligned} \tag{4.82}$$

This means that electric and magnetic field components in the **z** direction, the direction of propagation, must be constant with respect to **z**. Only the electric and magnetic field components transverse to the direction of propagation vary with respect to z. That is the electromagnetic wave is transverse in nature.

4.13 Polarization